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Duality invariance in massive theories

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In this work, we show that duality symmetry can be implemented for massive theories at the level of the action, whenever we can formulate appropriate gauge invariant actions. For a massive vectorial field, we use a known gauge invariant description, while for a massive graviton, we introduce a novel gauge invariant action in order to show duality invariance.

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I. INTRODUCTION

The duality symmetry has been one of the concepts most relevant in high-energy physics lately. In particular, the electromagnetic duality has played an important role since forty years ago, from the advent of supergravity to the recent results in superstring and M theory. The aspects of linear duality symmetries are well known, but the inclusion of sources and nonlinear generalizations (some few cases of interacting theories are known) open new challenges. Duality symmetry was initially understood at the level of the equations of motion [1]. The first successful attempt to establish electromagnetic duality symmetry at the level of the action was achieved in Ref. [2] after solving the Gauss constraint. The cost was the loss of explicit Lorentz invariance of the Maxwell action, when it is expressed in terms of transverse physical variables ($\partial_i \pi_i^T = 0 = \partial_i A_i^T$). The Hamiltonian first-order action

$$I = \int d^4x \left[\pi_i^T A_i^T - \frac{1}{2}(\vec{\pi}^T.\vec{\pi}^T + (\nabla \times \vec{A}^T)^2)\right] \qquad (1)$$

is invariant under the following (spatial nonlocal) duality transformations:

$$\delta\vec{\pi}^T = \nabla \times \vec{A}^T, \qquad \delta\vec{A}^T = \frac{1}{\nabla^2}\nabla \times \vec{E}^T. \qquad (2)$$

Since π_i^T satisfies the Gauss constraint, $\partial_i \pi_i^T = 0$, it is possible to introduce a second potential, $\pi_i^T = \epsilon_{ijk}\partial_j \tilde{A}_i^T$, to achieve the two potentials formulation of electromagnetism [3],

$$I = \int d^4x \left[\frac{1}{4}\epsilon^{ijk} F_{jk}^{\alpha} \mathcal{L}^{\alpha\beta} \dot{A}_i^{\beta} - \frac{1}{2} F_{ij}^{\alpha} F_{ij}^{\alpha}\right], \qquad (3)$$

where $A_i^\alpha \equiv (A_i, \tilde{A}_i)$, $F_{ij}^\alpha \equiv \partial_i A_j^\alpha - \partial_j A_i^\alpha$ (it is understood as the transverse character of the potentials), and

$$\mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (4)$$

An early approach with two potentials was proposed in Ref. [4], in which electric and magnetic sources were considered. Now, the action (3) is invariant under local duality transformations

$$A_i^\alpha \to \mathcal{L}^{\alpha\beta} A_i^\beta. \qquad (5)$$

These duality transformations and the action (3) boil down to Eq. (2) and the Maxwell action (in the gauge $A_o = 0$) after eliminating one of the potentials using the equations of motion corresponding to the action (3). Although the Lorentz invariance is not explicitly manifest, the action (3) has invariance under some transformations that are equivalent to the usual Lorentz transformations on shell [3]. This procedure was extended to the (nonlinear) Born–Infeld electrodynamics [5] and for antisymmetric fields in any dimensions [6]. Despite the difficulties in implementing duality invariance in Lorentz invariant action, there are two ways to circumvent this difficulty, namely, the introduction of infinite fields [7] and a new gauge field, called the PST field [8], which interact with the electromagnetic potentials in a nonpolynomial way, and it is equivalent to Eq. (3) after gauge fixing the PST field.

Remarkably enough is the achievement of duality symmetry for gravity [9] after solving the Hamiltonian and momentum constraints, which characterize the canonical formulation of the linearized gravity, by introducing two (pre)potentials. The action has the form

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$$I = \int dt d^3x [\epsilon^{ijk}\epsilon^{\alpha\beta}(\partial_{abj}Z^\alpha_{ak} - \nabla^2\partial_j Z^\alpha_{bj})\dot{Z}^\beta_{bi}] - \int dt H, \tag{6}$$

where $Z^\alpha_{ij} \equiv (\Phi_{ij}, P_{ij})$ are the prepotentials, which are related to the spatial components of the metric (h_{ij}) and the canonical momentum (π_{ij}) in a nonlocal way [9], and

$$H = \int d^3x \Big[\nabla^2 Z^\alpha_{ij}\nabla^2 Z^\alpha_{ij} + \frac{1}{2}\partial_{ij}Z^\alpha_{ij}\partial_{kl}Z^\alpha_{kl} + \partial_{ij}Z^\alpha_{ij}\nabla^2 Z^\alpha - 2\partial_{ik}Z^\alpha_{jk}\nabla^2 Z^\alpha_{ij} - \frac{1}{2}\nabla^2 Z^\alpha \nabla^2 Z^\alpha\Big] \tag{7}$$

is the Hamiltonian, and $Z^\alpha = \delta^{ij}Z^\alpha_{ij}$ are the traces of the prepotentials. This action is clearly invariant under duality transformations

$$Z^\alpha_{ij} \to \mathcal{L}^{\alpha\beta}Z^\beta_{ij}. \tag{8}$$

The generalization to arbitrary dimensions was achieved in Ref. [10], based on the duality relationship between the graviton field h_{mn} and the generalized Curtright field $T_{m_1\ldots m_{D-3},n}$ [11,12]. Afterward, the duality symmetric action for higher spin was completed in Ref. [13]. The essential point of this achievement is clearly expressed in this work: "The key to our derivation is the remark that since free gauge field actions are (abelian) gauge invariant, they can be uniformly expressed -after elimination of constraints- in terms of the fundamental spatial gauge-invariant symmetric transverse-traceless (TT) conjugate variables." The actions that are invariant under duality transformations for higher spin have the same structure as the electromagnetic duality invariant action (3). We will adopt this point of view when we deal with gauge invariant models for massive theories. For instance, if in the action (6) we consider the usual transverse-longitudinal decomposition for the (pre)potentials $Z^\alpha_{ij}(= Z^{\alpha TT}_{ij} + \partial_i Z^{\alpha T}_j + \partial_j Z^{\alpha T}_i + \partial_{ij}\alpha + \delta_{ij}\beta)$, it takes the form

$$I = \int d^4x \Big[\frac{1}{2}\epsilon^{ijk}\partial_j\nabla^2 Z^{TT\alpha}_{il}\mathcal{L}^{\alpha\beta}\dot{Z}^{TT\beta}_{kl} - \frac{1}{2}\nabla^2 Z^{TT\alpha}_{ij}\nabla^2 Z^{TT\alpha}_{ij}\Big]. \tag{9}$$

Moreover, the partially massless phenomenon for massive gravitons in (anti-)de Sitter [14], in which the mass of the graviton is fine-tuned to a value related to the cosmological constant, exhibits an electromagnetic duality invariance [15], in which a massive graviton acquires an additional gauge invariance due to the partially massless effect. Recently, duality properties of the Horava gravity [16], which has in common with Eq. (6) a higher-order spatial derivative Hamiltonian, were studied and discussed [17] in the framework of Ref. [12].

Another example of a duality invariant action (which will be used in this paper) is the case of an antisymmetric field (B_{mn}) of which the field strength is $H_{mnp} \equiv \partial_m B_{np} + \partial_n B_{pm} + \partial_p B_{mn}$. This is a particular case discussed in Ref. [6], in which the electromagnetic duality was considered for p forms in D dimensions. For an antisymmetric field, the canonical action is

$$I = \int d^4x \Big[\pi_{ij}\dot{B}_{ij} - \pi_{ij}\pi_{ij} - \frac{1}{12}H_{ijk}H_{ijk} + 2B_{oi}\mathcal{F}_i\Big], \tag{10}$$

where π_{ij} is the canonical momentum associated to B_{ij} and $\mathcal{F}_i \equiv \partial_j\pi_{ij} \simeq 0$ is the corresponding Gauss constraint of which the solution is $\pi_{ij} = -\frac{1}{2}\epsilon_{ijk}\partial_k\phi$. After decomposing the spatial antisymmetric field as $B_{ij} = \epsilon_{ijk}\frac{\partial_k}{\sqrt{-\nabla^2}}\psi + \partial_i b^T_j - \partial_j b^T_i$, the action will depend only on the dual variables (ϕ, ψ),

$$I = \int d^4x \Big[(\sqrt{-\nabla^2}\psi\dot{\phi} - \frac{1}{2}\partial_i\phi\partial_i\phi - \frac{1}{2}\partial_i\psi\partial_i\psi\Big], \tag{11}$$

which is invariant under $\phi \to \psi$ and $\psi \to -\phi$.

In summary, the method for obtaining duality invariant action relies on the fact of renouncing spacetime covariance in an explicit way. This does not mean breaking of Lorentz invariance. The Lorentz and duality symmetries have a closer connection [18] that was anticipated in Refs. [3] and [19]. Starting up with a Lorentz and gauge invariant action, the Hamiltonian formulation (described by a first-order action) is constructed, and the Gauss constraints are solved by introducing new potentials. These new sets of potentials (named electric and magnetic potentials) are the essential objects from which the new duality invariant action is constructed. In this paper, we generalize the duality invariant actions for massive vectorial (spin-1) and tensorial (spin-2) fields, which are described by gauge invariant actions that are obtained by dualization of the usual Stueckelberg formulations of these fields. In the next section, the massive gauge invariant vectorial model is considered, and we look for the duality properties in terms of the transverse and longitudinal degrees of freedom. Next, we construct a new gauge invariant action for the massive graviton from which, in the last section, we will establish the duality invariance at the level of the action. It is worth recalling that for massive theories there is no covariant duality relationship between field strengths and its dual, which interchanges field equations and Bianchi identities. Throughout this work, we use η_{mn} mostly positive and the convention of $\epsilon^{oijk} \equiv \epsilon^{ijk}$. We restrict our results to four dimensions.

II. DUALITY FOR MASSIVE SPIN 1

It is well known that the massive vectorial field can be described by a gauge invariant formulation that involves the coupling of the vectorial field A_m with an antisymmetric field B_{mn} through a topological term [20]. The action is

$$I = \int d^4x\left[-\frac{1}{4}F_{mn}F^{mn} - \frac{1}{12}H^{mnp}H_{mnp} - \frac{\mu}{4}\epsilon^{mnpq}B_{mn}F_{pq}\right], \tag{12}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$ and $H_{mnp} = \partial_m B_{np} + \partial_n B_{pm} + \partial_p B_{mn}$. This action is invariant under gauge transformations, $\delta A_m = \partial_m \lambda$ and $\delta B_{mn} = \partial_m \lambda_n - \partial_n \lambda_m$, and it is dual equivalent to the usual Stueckelberg formulation for massive spin 1 [21]. The field equations associated with Eq. (12) are

$$\partial_n\left(F^{nm} - \frac{\mu}{2}\epsilon^{mnpq}B_{pq}\right) = 0 \quad \text{and}$$
$$\partial_p(H^{pmn} - \mu\epsilon^{mnpq}A_q) = 0, \tag{13}$$

and the Bianchi identities are

$$\epsilon^{mnpq}\partial_n F_{pq} = 0 \quad \text{and} \quad \epsilon^{mnpq}\partial_q H_{mnp} = 0. \tag{14}$$

There are no duality transformations that exchange these covariant field equations with the Bianchi identities and vice versa. The goal will be to find an action in terms of prepotentials with invariance under local duality transformations. To reach a non-Lorentz invariant duality action, we must start with the Hamiltonian approach. The canonical momenta are

$$\pi^i = \frac{\delta L}{\delta \partial_o A_i} = F_{oi} - \frac{1}{2}\mu\epsilon_{ijk}B_{jk}$$
$$\pi^{ij} = \frac{\delta L}{\delta \partial_o B_{ij}} = \frac{1}{2}H_{oij}. \tag{15}$$

As usual, A_o and B_{oi} are multipliers. The action has the canonical form

$$I = \int d^4x[\pi^i \dot{A}_i + \pi^{ij}\dot{B}_{ij} - \mathcal{H} + A_o \mathrm{G} + B_{oi}\mathrm{G}^i], \tag{16}$$

where the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\left(\pi_i + \frac{1}{2}\mu\epsilon_{ijk}B^{jk}\right)\left(\pi^i + \frac{1}{2}\mu\epsilon^{iab}B_{ab}\right) + \frac{1}{4}F_{ij}F^{ij} + \pi_{ij}\pi^{ij} + \frac{1}{12}H_{ijk}H^{ijk} \tag{17}$$

and the Gauss constraints are

$$\mathrm{G} = \partial_i \pi^i = 0 \tag{18}$$

and

$$\mathrm{G}^i = \partial_j\left(\pi^{ij} + \frac{1}{2}\mu\epsilon^{ijk}A_k\right) = 0. \tag{19}$$

These constraints are easily solved (locally) for the momenta and lead to the "magnetic" potentials $\tilde{A}_i$ and ϕ,

$$\pi^i = \epsilon^{ijk}\partial_j \tilde{A}_k \tag{20}$$

and

$$\pi^{ij} + \frac{1}{2}\mu\epsilon^{ijk}A_k = \frac{1}{2}\epsilon^{ijk}\partial_k\phi. \tag{21}$$

When these solutions are considered, the action is written as

$$I = \int d^4x\left[\frac{1}{2}\epsilon^{ijk}\mu\left(B_{jk} + \frac{1}{\mu}\tilde{F}_{jk}\right)\left(\dot{A}_i - \frac{1}{\mu}\partial_i\dot{\phi}\right) - \frac{1}{2}(B_iB^i + H^2) - \frac{1}{2}\mu^2\left(A_i - \frac{1}{\mu}\partial_i\phi\right)^2 - \frac{1}{4}\mu^2\left(B_{ij} + \frac{1}{\mu}\tilde{F}_{ij}\right)^2\right], \tag{22}$$

where $\tilde{F}_{ij} = \partial_i\tilde{A}_j - \partial_j\tilde{A}_i$ and we have introduced the magnetic fields

$$B_i \equiv \frac{1}{2}\epsilon_{ijk}F_{jk} \quad H \equiv \frac{1}{6}\epsilon_{ijk}H_{ijk}. \tag{23}$$

At this stage, we have invariance under the gauge transformations

$$\delta A_i = \partial_i\lambda, \qquad \delta B_{ij} = \partial_i\lambda_j - \partial_j\lambda_i$$
$$\delta\tilde{A}_i = \partial_i\tilde{\lambda} - \mu\lambda_i, \qquad \delta\phi = \mu\lambda, \tag{24}$$

which allow us to fix the gauges $\phi = 0$ and $\tilde{A}_i = 0$. Then, we easily recognize $\frac{1}{2}\mu\epsilon^{ijk}B_{jk}$ as the canonical momentum associated to A_i, and the Hamiltonian first-order action for the massive vectorial field arises. But we need all the potentials that have emerged in order to have duality invariance in the action. We decompose the fields in its irreducible transverse and longitudinal components:

$$A_i = A_i^T + \partial_i A^L, \qquad B_{ij} = \epsilon_{ijk}\partial_k\psi + \partial_i b_j^T - \partial_j b_i^T. \tag{25}$$

The longitudinal component of the $\tilde{A}_k$ field is absent from the beginning [see Eq. (20)], while the longitudinal component of A_i is absorbed with the scalar field ϕ. The first term in the decomposition for the antisymmetric field B_{ij} is particular for $D = 4$; the scalar field ψ will be the dual partner of ϕ. Moreover, we define $a_i^T \equiv \frac{1}{\mu}A_i^T + b_i$ and its field strength $f_{ij} = \partial_i a_j - \partial_j a_i$. Then, the action is written out as (from now on, the transverse and longitudinal characters are understood)

$$I = \int d^4x \Big[\frac{1}{2}\mu\epsilon^{ijk} f_{jk}\dot{A}_i - \frac{1}{2}\mu^2 A_i A_i - \frac{1}{2}(F_{ij}F_{ij} + f_{ij}f_{ij}) + (\nabla^2\psi)\dot{\phi} - \frac{1}{2}\mu^2\partial_i\psi\partial_i\psi - \frac{1}{2}\partial_i\phi\partial_i\phi - \frac{1}{2}\nabla^2\psi\nabla^2\psi\Big]. \tag{26}$$

Thus, the pairs of dual partners (A_i, a_i) and (ϕ, ψ) are decoupled. Finally, we make the definitions

$$a_i = \frac{1}{\mu}\sqrt{1 - \frac{\mu^2}{\nabla^2}}\hat{A}_i, \qquad \phi = \sqrt{\mu^2 - \nabla^2}\varphi, \tag{27}$$

in order to arrive to the form of the action

$$I = \int d^4x \Big[\frac{1}{2}\sqrt{1 - \frac{\mu^2}{\nabla^2}}\epsilon^{ijk}\hat{F}_{jk}\dot{A}_i - \frac{1}{2}\mu^2 A_i A_i - \frac{1}{2}\mu^2\hat{A}_i\hat{A}_i - \frac{1}{2}(F_{ij}F_{ij} + \hat{F}_{ij}\hat{F}_{ij}) + (\nabla^2\psi)\sqrt{\mu^2 - \nabla^2}\dot{\varphi} - \frac{1}{2}\mu^2\partial_i\psi\partial_i\psi - \frac{1}{2}\mu^2\partial_i\varphi\partial_i\varphi - \frac{1}{2}\nabla^2\varphi\nabla^2\varphi - \frac{1}{2}\nabla^2\psi\nabla^2\psi\Big] \tag{28}$$

or

$$I = \int d^4x \Big[\frac{1}{4}\sqrt{1 - \frac{\mu^2}{\nabla^2}}\epsilon^{ijk} F^\beta_{jk}\mathcal{L}^{\alpha\beta}\dot{A}^\alpha_i - \frac{1}{2}\mu^2 A^\alpha_i A^\alpha_i - \frac{1}{2}F^\alpha_{ij}F^\alpha_{ij} + \frac{1}{2}(\nabla^2\Phi^\beta)\sqrt{\mu^2 - \nabla^2}\mathcal{L}^{\alpha\beta}\dot{\Phi}^\alpha - \frac{1}{2}\mu^2\partial_i\Phi^\alpha\partial_i\Phi^\alpha - \frac{1}{2}\nabla^2\Phi^\alpha\nabla^2\Phi^\alpha\Big], \tag{29}$$

where we have introduced the conventional notation: $A^\alpha_i = (A_i, \hat{A}_i)$, $\Phi^\alpha = (\varphi, \psi)$. The action is clearly invariant under duality transformations

$$A^\alpha_i \to \mathcal{L}^{\alpha\beta} A^\beta_i \quad \text{and} \quad \Phi^\alpha \to \mathcal{L}^{\alpha\beta}\Phi^\beta. \tag{30}$$

The field equations obtained from Eq. (29) are

$$\sqrt{1 - \frac{\mu^2}{\nabla^2}}\epsilon^{ijk}\mathcal{L}^{\alpha\beta}\partial_j \dot{A}^\beta_i + (\mu^2 - \nabla^2)A^\alpha_i = 0 \tag{31}$$

and

$$\sqrt{\mu^2 - \nabla^2}\mathcal{L}^{\alpha\beta}\dot{\Phi}^\beta - (\mu^2 - \nabla^2)\Phi^\alpha = 0. \tag{32}$$

By iteration of these first-order differential equations, it is easily checked that the potentials A^α_i and Φ^α satisfy the Klein–Gordon equation:

$$(\Box - \mu^2)A^\alpha_i = 0 \quad \text{and} \quad (\Box - \mu^2)\Phi^\alpha = 0. \tag{33}$$

The canonical momenta are slightly modified by the presence of a mass term, and the second-class constraints are derived,

$$\Upsilon^\alpha_i \equiv \pi^\alpha_i - \frac{1}{4}\sqrt{1 - \frac{\mu^2}{\nabla^2}}\epsilon^{ijk}F^\beta_{jk}\mathcal{L}^{\alpha\beta} \approx 0,$$
$$\Gamma^\alpha \equiv \pi^\alpha - \frac{1}{2}\sqrt{\mu^2 - \nabla^2}(\nabla^2\Phi^\beta)\mathcal{L}^{\alpha\beta} \approx 0, \tag{34}$$

from which the Poisson bracket for these second-class constraints now reads

$$[\Upsilon^\alpha_{i(x)}, \Upsilon^\beta_{j(y)}] = \sqrt{1 - \frac{\mu^2}{\nabla^2}}\mathcal{L}^{\alpha\beta}\epsilon_{ijk}\partial_k\delta_{(x-y)},$$
$$[\Gamma^\alpha_{(x)}, \Gamma^\beta_{(y)}] = \sqrt{\mu^2 - \nabla^2}\mathcal{L}^{\alpha\beta}\nabla^2\delta_{(x-y)}. \tag{35}$$

These momenta must transform under duality as $\pi^\alpha_i \to \mathcal{L}^{\alpha\beta}\pi^\beta_i$ and $\pi^\alpha \to \mathcal{L}^{\alpha\beta}\Phi^\pi$ in order to keep the invariance of the Poisson bracket under duality. The action (29) is invariant under the global transformations

$$\delta A^\alpha_i = x^o v^j \partial_j A^\alpha_i + \sqrt{1 - \frac{\mu^2}{\nabla^2}}(\vec{v}.\vec{x})\epsilon^{ijk}\mathcal{L}^{\alpha\beta}\partial_j A^\beta_k \tag{36}$$

and

$$\delta\Phi^\alpha = x^o v^j \partial_j \Phi^\alpha + \sqrt{\mu^2 - \nabla^2}(\vec{v}.\vec{x})\mathcal{L}^{\alpha\beta}\nabla^2\Phi^\beta, \tag{37}$$

where $\vec{v}$ is an arbitrary constant three-dimensional vector. If we use the solutions of the constraints (34), these become the usual Lorentz boosts transformations

$$\delta\mathbf{X} = x^o v^j \partial_j \mathbf{X} + (\vec{v}.\vec{x})\partial_o \mathbf{X}, \tag{38}$$

where $\mathbf{X} = (A_i^1, \phi)$ and we have considered that $2\pi_i^1 = \partial_o A_i^1$ and $2\pi^1 = \partial_o \phi$. Moreover, the action (29) is manifestly invariant under rotations.

In the limit $\mu \to 0$, the action (29) is the sum of the actions (3) and (11), which indicate that there is no discontinuity of the number of degrees of freedom as can be seen from the covariant Lorentz action (12), where the scalar field, which provides mass to the vectorial field and is represented by the antisymmetric field B_{mn}, is cleanly decoupled from the vectorial action in this limit.

III. GAUGE INVARIANT FORMULATION FOR THE MASSIVE GRAVITY

For massive gravity, we need a new gauge invariant formulation for the symmetric field (h_{mn}), which generalizes the action (12) for massive spin 2. To construct this action, we start with the usual Stueckelberg formulation for the Fierz–Pauli action,

$$I = I_{\text{lin}} + \int d^4x \Big[-\frac{1}{4}\mu^2 (h_{mn} + \partial_m a_n + \partial_n a_m)^2 + \frac{1}{4}\mu^2 (h + 2\partial_m a_m)^2 \Big], \tag{39}$$

where

$$I_{\text{lin}} = \int d^4x \Big[-\frac{1}{4}\partial_p h_{mn}\partial_p h^{mn} + \frac{1}{4}\partial_p h \partial_p h + \frac{1}{2}\partial_n h_{mn}\partial_p h^{mp} - \frac{1}{2}\partial_m h \partial_n h^{mn} \Big] \tag{40}$$

is the linearized Einstein action and a_m is the vectorial Stueckelberg field that guarantee gauge invariance under the following transformations:

$$\delta h_{mn} = \partial_m \xi_n + \partial_n \xi_m, \qquad \delta a_m = -\xi_m. \tag{41}$$

We can dualize this action. For this process, we substitute $\partial_m a_n + \partial_n a_m$ by a symmetric tensor g_{mn} and by introducing a new term into the action that enforces that linearized curvature tensor $R_{mnpq(g)} (\equiv \partial_{np} g_{mq} + \partial_{mq} g_{np} - \partial_{nq} g_{mp} - \partial_{mp} g_{nq})$ vanishes through a Lagrange multiplier B_{mnpq} and that has the same symmetries as the linearized curvature tensor R_{mnpq}. Then, we have the following action:

$$I = I_{\text{lin}} + \int d^4x \Big[-\frac{1}{4}\mu^2 (h_{mn} + g_{mn})(h^{mn} + g^{mn}) + \frac{1}{4}\mu^2 (h + g)^2 + \frac{1}{8} B^{mnpq} R_{mnpq(g)} \Big]. \tag{42}$$

After considering the constraint imposed by the multiplier B_{mnpq}, which tells us that $g_{mn} = \partial_m a_n + \partial_n a_m$, we obtain Eq. (39). On the other hand, we can determine g_{mn} by using its field equation

$$g_{mn} = -h_{mn} - \frac{1}{\mu^2}\Big(\partial_p \partial_q B_{mpnq} - \frac{1}{3}\eta_{mn}\partial_p\partial_q B_{rprq} \Big) \equiv -h_{mn} + f_{mn}, \tag{43}$$

where we have defined $f_{mn} = -\frac{1}{\mu^2}(\partial_p\partial_q B_{mpnq} - \frac{1}{3}\eta_{mn}\partial_p\partial_q B_{rprq})$. After introducing Eq. (43) into the action (42) (and redefining $B_{mnpq} \to \mu B_{mnpq}$), we reach the result

$$I = \int d^4x \Big[-\tfrac{1}{4}\partial_p h_{mn}\partial_p h^{mn} + \tfrac{1}{4}\partial_p h \partial_p h + \tfrac{1}{2}\partial_n h_{mn}\partial_p h^{mp} - \frac{1}{2} h \partial_m \partial_n h^{mn} \Big] - \frac{1}{8}\mu \int d^4x B^{mnpq} R_{mnpq(h)} - \frac{1}{8}\mu \int d^4x B^{mnpq} R_{mnpq(f)} - \frac{1}{4}\mu^2 \int d^4x (f_{mn} f^{mn} - f^2), \tag{44}$$

and this action expressed only in terms of the fields h_{mn} and B_{mnpq} is written as

$$I = I_{\text{lin}} + I_{\text{DST}} - \frac{\mu}{8}\int d^4x B^{mnpq} R_{mnpq(h)}, \tag{45}$$

where

$$I_{\text{DST}} = \frac{1}{4}\int d^4x \Big[\partial_p \partial_q B_{mpnq} \partial_r \partial_s B_{mrns} - \frac{1}{3}(\partial_p\partial_q B_{npnq})^2 \Big] \tag{46}$$

is the free ghost higher-derivative action found by Deser *et al.* [22] and which is an alternative description of the Maxwell action. In the Appendix, some aspects of this action are reviewed. The action (45) is invariant under the gauge transformations

$$\delta h_{mn} = \partial_m \xi_n + \partial_n \xi_m, \qquad \delta B^{mnpq} = \epsilon^{mnrs}\partial_r \lambda^{pq},_s + (mn) \leftrightarrow (pq), \tag{47}$$

where the gauge parameters are antisymmetric in the pair (pq), i.e., $\lambda^{pq},_s = -\lambda^{qp},_s$ without any additional properties in its indices. The action (46) has a conformal invariance that is lost here by the coupling term $\sim B^{mnpq} R_{mnpq(h)}$.

The action (44) is our gauge invariant formulation for massive spin 2 in which we consider h_{mn}, B_{mnpq}, and f_{mn} as independent fields. The field f_{mn} is an auxiliary field. This way of describing the field B^{mnpq} with the use of an auxiliary field f^{mn} was introduced in Ref. [22]. The coefficients in this action are the same for any dimensions, while the coefficient $\frac{1}{3}$ in the second term of Eq. (46) must be replaced by $\frac{1}{D-1}$. In particular, in three dimensions, the action (44) becomes the action for the new massive gravity [23]. In fact, in three dimensions, we have the identity $R_{mnpq} = \epsilon_{mnr}\epsilon_{pqs}G^{rs}$, which tells us that the curvature is completely determined by the Einstein tensor G_{mn} since the conformal tensor vanishes identically. Then, we rewrite the term $-\frac{1}{8}B^{mnpq}R_{mnpq(h+f)}$ as $-\frac{1}{2}B^{mn}G_{mn(h+f)}$, where we have defined $B^{mn} \equiv \epsilon^{mpq}\epsilon^{nrs}R_{pqrs}$, and the action can be rewritten as

$$I_{3D} = \int d^3x \left[\frac{1}{4}h^{mn}G_{mn} - \frac{1}{2}B^{mn}G_{mn(h+f)} - \frac{1}{2}\mu^2(f^{mn}f_{mn} - f^2)\right], \tag{48}$$

which becomes the action for the new massive gravity after eliminating the h_{mn} field through its field equation.

For the sake of completeness, let us sketch the dynamical content of the action (44) to confirm that our model propagates the 5 degrees of freedom of the massive spin 2 in four dimensions. The temporal and spatial components are denoted by $h_{mn}\colon (h_{oo} \equiv \psi, h_{oi}, h_{ij})$, $f_{mn}\colon (f_{oo} \equiv \theta, f_{oi}, f_{ij})$, and $(S_{ij} \equiv B_{iooj} = S_{ji}, W_{ij} \equiv \partial_k(B_{oikj} + B_{ojki}) = W_{ji}, V_{ij} \equiv \partial_{kl}B_{iklj} = V_{ji})$. When the action is written out in terms of these components, the B_{ijkl} variable appears as a Lagrange multiplier associated with the constraint $R_{ijkl(h+f)} = 0$, which can be solved (locally) as $f_{ij} = h_{ij} + \partial_i\kappa_j + \partial_j\kappa_i$. We can use the gauge invariance (47) in order to fix the gauge $\kappa_i = 0$. Now, we consider the usual transverse and longitudinal decompositions,

$$\begin{aligned} h_{ij} &= \chi_{ij} + \partial_i h_j + \partial_j h_i + \partial_{ij}\sigma + \delta_{ij}\tau, \\ h_{oj} &= u_i + \partial_i v, \end{aligned} \tag{49}$$

$$f_{ij} = \tau_{ij} + \partial_i f_j + \partial_j f_i + \partial_{ij}\lambda + \delta_{ij}\alpha, \qquad f_{oj} = r_i + \partial_i t, \tag{50}$$

and

$$\begin{aligned} S_{ij} &= s_{ij} + \partial_i s_j + \partial_j s_i + \partial_{ij}s + \delta_{ij}\beta, \\ W_{ij} &= w_{ij} + \partial_i w_j + \partial_j w_i + \partial_{ij}w + \delta_{ij}\gamma, \end{aligned} \tag{51}$$

where $\chi_{ij}, \tau_{ij}, s_{ij}$, and w_{ij} are transverse and traceless tensor; h_j, u_i, f_i, r_i, s_i, and w_i are transverse vectors; and $\sigma, \tau, v, \lambda, \alpha, t, s, \beta, w$, and γ are scalars. The solution of the constraint imposed by the B_{ijkl} tells us

$$\tau_{ij} = -\chi_{ij}, \qquad f_i = -h_i, \qquad \alpha = -\tau, \qquad \lambda = -\sigma. \tag{52}$$

It is easily checked that s_{ij} and w_{ij} do not enter into the action; then the transverse and traceless sector of the action is, as it must be,

$$\frac{1}{4}\chi_{ij}\Box\chi_{ij} - \frac{1}{4}\mu^2\chi_{ij}\chi_{ij}, \tag{53}$$

which represents the propagation of the 2 degrees of freedom of the tensorial sector The vectorial sector of the action is

$$\begin{aligned} &-u_i\nabla^2 u_i + 2u_i\nabla^2\dot{h}_i - \dot{h}_i\nabla^2\dot{h}_i + 2\mu u_i\nabla^2\dot{s}_i + 2\mu u_i\nabla^2 w_i \\ &\quad + 2\mu r_i\nabla^2\dot{s}_i + 2\mu r_i\nabla^2 w_i + \mu^2 r_i r_i + \mu^2 h_i\nabla^2 h_i, \end{aligned} \tag{54}$$

and it is clear that u_i and r_i are auxiliary fields that are determined using its field equations. Its values are

$$u_i = \dot{h}_i + \mu\dot{s}_i + \mu w_i, \qquad r_i = -\frac{\nabla^2}{\mu}\dot{s}_i - \frac{\nabla^2}{\mu}w_i, \tag{55}$$

and after substituting (55) into the vectorial sector of the action, w arises as un auxiliary field and can be determined through its field equation

$$w_i = -\mu\dot{s}_i - \frac{\mu}{\mu^2 - \nabla^2}\dot{h}_i. \tag{56}$$

We can see that, after taking into account (56), the vectorial sector of the action depends on h_i only, and redefining $h_i = \frac{1}{\mu}\sqrt{1 - \frac{\mu^2}{\nabla^2}}\bar{h}_i$, we end with the propagation of the 2 degrees of freedom of the vectorial sector

$$\sim \bar{h}_i\Box\bar{h}_i - \mu^2\bar{h}_i\bar{h}_i. \tag{57}$$

In the scalar sector, we found that θ, γ, t, and w are not dynamical variables (these are multipliers or are solved), and this sector reduced to the scalar sector of the Fierz–Pauli action:

$$\begin{aligned} &2\psi\nabla^2\tau + 4v\nabla^2\dot{\tau} + 4\sigma\nabla^2\ddot{\tau} + 3\tau\ddot{\tau} - \tau\nabla^2\tau + 2\mu^2\sigma\nabla^2\tau \\ &\quad + 3\mu^2\tau\nabla^2\tau - \mu^2 v\nabla^2 v - \mu^2\psi\nabla^2\sigma - 3\mu^2\psi\tau. \end{aligned} \tag{58}$$

Clearly, ψ is a multiplier, and its corresponding constraint is $2\nabla^2\tau - \mu^2\nabla^2\sigma - 3\mu^2\tau = 0$, which is solved for σ $(= \frac{2}{\mu^2}\tau - \frac{3}{\nabla^2}\tau)$, and after introducing this value into the scalar sector, the v variable appears quadratically without temporal derivatives, and its field equation determines it:

$v = \frac{2}{\mu^2}\dot{\tau}$. With this result at hand, we reach the final unconstrained form of the propagation of 1 scalar degree of freedom:

$$\sim\tau(\Box - \mu^2)\tau. \tag{59}$$

Thus, we have seen that our action for describing in a gauge invariant way the massive graviton propagates the usual 5 free ghost degrees of freedom.

IV. DUALITY FOR MASSIVE GRAVITY

In this section, we show that the unconstrained Hamiltonian form of the gauge invariant action (44) admits the introduction of new potentials in order to reach a duality invariance for the massive spin 2. To begin with, we need a first-order canonical action of (44). The canonical momenta are

$$\pi_{ij} = \frac{\delta I}{\delta \dot{h}_{ij}} = \frac{1}{2}\dot{h}_{ij} - \frac{1}{2}\delta_{ij}\dot{h} - \frac{1}{2}(\partial_i h_{oj} + \partial_j h_{oi}) + \delta_{ij}\partial_k h_{ok} + \frac{1}{2}\mu b_{ij}, \tag{60}$$

where h is the spatial trace of h_{ij} and we have defined $b_{ij} = \dot{S}_{ij} + W_{ij}$,

$$\pi_i = \frac{\delta I}{\delta \dot{h}_{oi}} = 0, \quad \sigma_{ij} = \frac{\delta I}{\delta \dot{f}_{ij}} = \frac{1}{2}\mu b_{ij}, \quad \sigma_i = \frac{\delta I}{\delta \dot{f}_{oi}} = 0. \tag{61}$$

The first-order canonical action is

$$I\int d^4x\left[\pi_{ij}\dot{h}_{ij} + \frac{1}{2}\mu b_{ij}\dot{f}_{ij} - \mathbb{H} - n_i\mathcal{H}_i - n\mathcal{H}\right], \tag{62}$$

where n_i and n are the shift and lapse functions associated with the momentum ($\mathcal{H}_i \approx 0$) and Hamiltonian ($\mathcal{H}_i \approx 0$) constraints, respectively,

$$\mathcal{H}_i \equiv \partial_j\pi_{ij} \approx 0, \qquad \mathcal{H} \equiv \nabla^2 h - \partial_{ij}h_{ij} - \mu\partial_{ij}S_{ij} \approx 0. \tag{63}$$

The Hamiltonian constraint is modified by the presence of the mass term in the action, while the momentum constraint remains intact.

The Hamiltonian density is

$$\mathbb{H} = \left[\pi_{ij} - \frac{1}{2}\mu b_{ij}\right]\left[\pi^{ij} - \frac{1}{2}\mu b^{ij}\right] - \left[\pi - \frac{1}{2}\mu b\right]^2 + \mathcal{R} + \frac{1}{2}\mu(h_{ij} + f_{ij})V_{ij} - \mu(\partial_j b_{ij})f_{oj} - \frac{1}{2}\mu^2 f_{oi}f_{oi} + \frac{1}{4}\mu^2 f_{ij}f_{ij} - \frac{1}{4}\mu^2 f_{ii}f_{jj} + \frac{1}{2}f_{oo}(\mu S_{ij} + \mu^2 f_{ii}), \tag{64}$$

with

$$\mathcal{R} \equiv \frac{1}{4}\partial_k h_{ij}\partial_k h^{ij} - \frac{1}{4}\partial_k h\partial_k h - \frac{1}{2}\partial_k h_{ki}\partial_j h^{ji} + \frac{1}{2}\partial_i h\partial_j h^{ji}. \tag{65}$$

Note that f_{oo} and B_{ijkl} are Lagrange multipliers associated with the constraints: $\partial_{ij}S_{ij} + \mu f_{ii} = 0$ and $R_{ijkl(h+f)} = 0$. The former allow us express the double spatial derivatives of S_{ij} in terms of the spatial trace of f_{ij}, while the latter tell us that $f_{ij} = -h_{ij} + \partial_i a_j + \partial_j a_i$, and since we have gauge invariance, one can set $a_i = 0$. Also, f_{oi} is an auxiliary field, and its field equation allows us to determine it, $f_{oi} = -\frac{1}{\mu}\partial_j b_{ij}$, and considering this value, the term $-\frac{1}{2}\partial_j b_{ij}\partial_k b_{ik}$ will emerge in the action. Now, we proceed to resolve the constraints by introducing the prepotentials [9]. The solution for the momentum constraint ($\mathcal{H}_i \approx 0$) is the same as the massless case

$$\pi_{ij} = \epsilon_{ika}\epsilon_{jlb}\partial_{kl}\tilde{P}_{ab}, \tag{66}$$

where $\tilde{P}_{ij}$ is the symmetric prepotential associated with the momentum constraint. To solve the Hamiltonian constraint ($\mathcal{H} \approx 0$), we decompose the six components of the spatial metric in its traceless part $\hat{h}_{ij}$ (five components) and its trace h (one component) as

$$h_{ij} = \hat{h}_{ij} + \frac{1}{3}\delta_{ij}h. \tag{67}$$

After considering this decomposition in the Hamiltonian constraint, we see that the traceless part must satisfy the following differential equation:

$$\partial_{ij}\hat{h}_{ij} - \frac{2}{3}\nabla^2 h + \mu^2 h = 0. \tag{68}$$

The solution is

$$\hat{h}_{ij} = J_{ij} + \left(\frac{\partial_{ij}}{\nabla^2} - \frac{1}{3}\delta_{ij}\right)\left(1 - \frac{3\mu^2}{\nabla^2}\right)h, \tag{69}$$

where (as introduced in Ref. [9])

$$J_{ij} = \epsilon_{ikl}\partial_k\Phi_{lj} + \epsilon_{jkl}\partial_k\Phi_{li}, \tag{70}$$

with Φ_{ij} being the symmetric prepotential linked to the Hamiltonian constraint. Essentially, these prepotentials will describe the transverse and traceless spin-2 sector of our massive model like the massless case. Before considering these prepotentials in the action, we redefine $\tilde{P}_{ij}$:

$$\tilde{P}_{ij} = P_{ij} - \frac{1}{2}\frac{\mu}{\nabla^2} b_{ij} + \frac{1}{2}\frac{\mu}{\nabla^2}\delta_{ij} b. \tag{71}$$

With these results, the first-order canonical action is written out as

$$I = \int d^3x dt \Bigg[\left(\pi_{ij(P)} - \frac{1}{2}\frac{\mu}{\nabla^2}(\partial_{ik} b_{jk} + \partial_{jk} b_{ik}) + \frac{1}{2}\frac{\mu}{\nabla^2}\delta_{ij}\partial_{kl} b_{kl}\right)\dot{J}_{ij(\Phi)} - \mathcal{R}_{(\Phi,h)} + r_{(\Phi)} - \pi_{ij(P)}\pi_{ij(P)} + \frac{1}{2}\pi^2_{(P)} - \frac{1}{2}\left(1 - \frac{\mu^2}{\nabla^2}\right)(\partial_j b_{ij})(\partial_k b_{ik})\Bigg], \tag{72}$$

where

$$\mathcal{R}_{(\Phi,h)} = -\frac{1}{4} J_{ij}\nabla^2 J_{ij} + \frac{1}{8}\frac{\mu^4}{\nabla^2} h^2 \tag{73}$$

and

$$r_{(\Phi)} = -\mu^2 \partial_k \Phi_{ij}\partial_k \Phi_{ij} + \frac{3}{2}\mu^2 \partial_j \Phi_{ij}\partial_k \Phi_{ik} + \mu^2 \Phi \partial_{ij}\Phi_{ij} - \frac{1}{2}\mu^2 \Phi \nabla^2 \Phi. \tag{74}$$

Now, we consider the different irreducible pieces of the involved variables, e.g., $\Phi_{ij} = \Phi^{TT}_{ij} + \partial_i \phi^T_j + \partial_j \phi^T_i + \partial_{ij}\phi + \delta_{ij}\rho$, and for $P_{ij} = P^{TT}_{ij} + \partial_i p^T_j + \partial_j p^T_i + \partial_{ij}\sigma + \delta_{ij}\tau$. The superscripts TT and T mark the transverse-traceless and transverse characters of the variable, respectively. Note that b_{ij} appears only as a vector, $\partial_j b_{ij}$, which we named b_i $(= b^T_i + \partial_i b)$. The total action is written out as

$$I = I^{TT}_2 + I^T_1 + I_0. \tag{75}$$

The transverse-traceless sector of the action boils down to [after making $(P, \Phi) \to \frac{1}{\sqrt{-\nabla^2}}(P, \Phi)$]

$$I^{TT}_2 = \int d^3x dt \left[\epsilon_{ijk}(\partial_j P^{TT}_{kl})\dot{\Phi}^{TT}_{il} - \frac{1}{2}\partial_k \Phi^{TT}_{ij}\partial_k \Phi^{TT}_{ij} - \frac{1}{2}\partial_k P^{TT}_{ij}\partial_k P^{TT}_{ij} - \frac{1}{2}\mu^2 \Phi^{TT}_{ij}\Phi^{TT}_{ij}\right], \tag{76}$$

and redefining $P^{TT}_{ij} = \sqrt{1 - \frac{\mu^2}{\nabla^2}}\hat{P}^{TT}_{ij}$, we obtain

$$I^{TT}_2 = \int d^3x dt \Bigg[\sqrt{1 - \frac{\mu^2}{\nabla^2}}\epsilon_{ijk}(\partial_j \hat{P}^{TT}_{kl})\dot{\Phi}^{TT}_{il} - \frac{1}{2}\partial_k \Phi^{TT}_{ij}\partial_k \Phi^{TT}_{ij} - \frac{1}{2}\partial_k P^{TT}_{ij}\partial_k P^{TT}_{ij} - \frac{1}{2}\mu^2 \Phi^{TT}_{ij}\Phi^{TT}_{ij} - \frac{1}{2}\mu^2 \hat{P}^{TT}_{ij}\hat{P}^{TT}_{ij}\Bigg], \tag{77}$$

which is clearly invariant under duality transformation,

$$\Phi^{TT}_{ij} \to \hat{P}^{TT}_{ij} \quad \text{and} \quad \hat{P}^{TT}_{ij} \to -\Phi^{TT}_{ij}. \tag{78}$$

Next, we move on the vectorial sector, which is described by

$$I^T_1 = \int d^3x dt \left[\mu b^T_i \epsilon_{ijk}\partial_j \dot{\phi}^T_k - \frac{1}{2} b^T_i \left(1 - \frac{\mu^2}{\nabla^2}\right) b^T_i - \frac{1}{2}\mu^2 \nabla^2 \phi^T_i \nabla^2 \phi^T_{ij}\right]; \tag{79}$$

then, after making some redefinitions on b^T_i and ϕ^T_i, we reach the form for the transverse vectorial sector of our action

$$I^T_1 = \int d^3x dt \left[\sqrt{1 - \frac{\mu^2}{\nabla^2}}\epsilon_{ijk}(\partial_j b^T_k)\dot{\phi}^T_i - \frac{1}{2}\partial_k b^T_i \partial_k b^T_i - \frac{1}{2}\partial_k \phi^T_i \partial_k \phi^T_i - \frac{1}{2}\mu^2 \phi^T_i \phi^T_i - \frac{1}{2}\mu^2 b^T_i b^T_i\right], \tag{80}$$

which is clearly invariant under duality transformations

$$\phi^T_i \to b^T_i \quad \text{and} \quad b^T_i \to -\phi^T_i. \tag{81}$$

Finally, we have in principle three scalar variables, τ (from the trace of $\pi_{ij(P)}$), b, and h, in the scalar sector of the action

$$I_0 = \int d^3x dt \left[\mu^2 \tau \dot{h} - \mu \dot{h}\left(\frac{1}{2} - \frac{\mu^2}{\nabla^2}\right) b + \frac{3}{8}\frac{\mu^4}{\nabla^2} h \left(1 - \frac{\mu^2}{\nabla^2}\right) h - \mu(\nabla^2 b)\tau + \frac{1}{2} b \left(\nabla^2 - \frac{5}{4}\mu^2\right) b\right]; \tag{82}$$

then we make the (canonical) identification $\mu^2 \tau - \mu(\frac{1}{2} - \frac{\mu^2}{\nabla^2}) b \equiv \frac{3}{8}\frac{\mu^3}{\nabla^2} b$ in order to write the scalar action as (redefining $h \to \frac{\nabla^2}{\mu} h$)

$$I_0 = \frac{3\mu^2}{4}\int d^3x dt \left[b\dot{h} - \frac{1}{2} h(\mu^2 - \nabla^2) h - \frac{1}{2}\mu^2 b^2\right], \tag{83}$$

and defining $b \equiv \sqrt{1 - \frac{\nabla^2}{\mu^2}}\tilde{b}$, we arrive at

$$I_0 = \frac{3\mu^2}{4}\int d^3x dt \left[\sqrt{\mu^2 - \nabla^2}\tilde{b}\,\dot{h} - \frac{1}{2}h(\mu^2 - \nabla^2)h - \frac{1}{2}\tilde{b}(\mu^2 - \nabla^2)\tilde{b} \right], \tag{84}$$

clearly invariant under duality transformations:

$$h \to \tilde{b} \quad \text{and} \quad \tilde{b} \to -h. \tag{85}$$

We have established duality invariance of massive gravity through the three levels given in the action (75). The first level is achieved with the same dual partners as the massless case, Φ_{ij} and P_{ij}, while in the other two layers, the dual partners involved the lower-spin components of h_{mn} and B_{mnpq}. In the massless limit, our result results tend to the decoupling of the three terms with the same structure as Eqs. (3), (9), and (11).

V. CONCLUSIONS

We have seen that if we deal with adequate gauge invariant formulations for massive theories it is possible to reach unconstrained duality invariant actions by using the method developed in Refs. [2,9], and [6]: to wit, start with a Hamiltonian gauge invariant first-order action, and solve the gauge constraints by introducing the named magnetic potentials that, together with the initial electric potentials, are the basic entities to establish duality invariance. These actions involved new fields that are coupled through topological terms. We have explicitly constructed this kind of action for massive gravitons (44). One must expect that it is not possible to have a successful coupling with a gravitational background because the action of Deser *et al.* has a serious problem with this kind of coupling [22]. But one can attempt to reach a nonlinear version of Eq. (44) like the Freedman–Townsend theory [24]. It would be interesting to make an exhaustive analysis of the action (44), in particular its properties in the massless limit and when a source is included.

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APPENDIX: ACTION OF DESER–SIEGEL–TOWNSEND

The action of Deser–Siegel–Townsend (46) is rewritten out as

$$I_{\mathrm{DST}} = \frac{1}{4}\int d^4x \left[W_{mn}W^{mn} - \frac{1}{3}W^2 \right], \tag{A1}$$

where $W_{mn} \equiv -\partial_{pq}B_{mpnq} = W_{nm}$ and W is its trace. In this form, this action has the same form as introduced in Ref. [25], in which W_{mn} is considered an independent field and subject to the constraint $\partial_n W^{mn} = 0$. If we take into account this constraint, we must attempt to consider the following action:

$$I = \frac{1}{4}\int d^4x \left[W_{mn}W^{mn} - \frac{1}{3}W^2 - W^{mn}(\partial_m a_n + \partial_n a_m) \right]. \tag{A2}$$

Here, W_{mn} and a_m are independent fields. This action was obtained by dimensional reduction in Ref. [26]. If one looks the a_m field as a multiplier, the associated constraint is just $\partial_n W^{mn} = 0$, the local solution of which introduces the B^{mpqn} field ($W^{mn} = \partial_{pq}B^{mpqn}$), and the fourth-order action (A1) is obtained. On the other hand, independent variations over W^{mn} lead to determining it, $W^{mn} = -(\partial_m a_n + \partial_n a_m) + 2\eta_{mn}(\partial.a)$, and substituting back into Eq. (A2), the Maxwell action arises $[I_{\mathrm{maxwell}} \sim -(\partial_m a_n - \partial_n a_m)^2]$. The inclusion of a mass term into Eq. (A2) was achieved in Ref. [27].

Originally, this equivalence was shown through the following second-order action (46):

$$I = \int d^4x \left[\frac{1}{8}\int d^4x B^{mnpq}R_{mnpq(f)} - \frac{1}{4}(f_{mn}f^{mn} - f^2) \right]. \tag{A3}$$

The fourth-order action (A1) has the gauge transformations

$$a)\, \delta B^{mnpq} = \epsilon^{mnrs}\partial_r \lambda^{pq}{}_{,s} + (mn) \leftrightarrow (pq) \tag{A4}$$

and a conformal transformation

$$b)\, \delta B^{mnpq} = (\eta_{mp}\eta_{nq} - \eta_{np}\eta_{mq})\varrho. \tag{A5}$$

This last invariance is not present in our gauge invariant description for massive spin 2, due to the coupling term (45).

If the gauge parameters $\lambda_{mn,p}$ are decomposed in irreducible pieces as

$$\lambda_{mn,p} = t_{mn,p} + c_{mnp}, \tag{A6}$$

where $t_{mn,p}$ satisfies the cyclic identity, $t_{[mn,p]} \equiv 0$, and c_{mnp} is completely antisymmetric, we can write that $t_{mn,p} = \partial_m \omega_{np} - \partial_n \omega_{mp}$ with $\omega_{mn} = \omega_{nm}$ (symmetric) and $c_{mnp} = \epsilon_{mnpq}\xi^q$, and then Eq. (A4) leads to the two gauge symmetries presented in Ref. [22].

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